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Quantum hypercomputation based on the dynamical algebra $\mathfrak{su}(1,1)$

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Abstract

An adaptation of Kieu's hypercomputational quantum algorithm (KHQA) is presented. The method that was used was to replace the Weyl–Heisenberg algebra by other dynamical algebra of low dimension that admits infinite-dimensional irreducible representations with naturally defined generalized coherent states. We have selected the Lie algebra $\mathfrak{su}(1,1)$, because this algebra possesses the necessary characteristics to realize the hypercomputation and also because such algebra has been identified as the dynamical algebra associated with many relatively simple quantum systems. In addition to an algebraic adaptation of KHQA over the algebra $\mathfrak{su}(1,1)$, we presented an adaptation of KHQA over some concrete physical referents: the infinite square well, the infinite cylindrical well, the perturbed infinite cylindrical well, the Pöschl–Teller potentials, the Holstein–Primakoff system and the Laguerre oscillator. We conclude that it is possible to have many physical systems within condensed matter and quantum optics in which it is possible to consider an implementation of KHQA.

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1. Introduction

Hypercomputers compute functions or numbers, or more generally solve problems or carry out tasks, that cannot be computed or solved by a Turing machine (TM) [1, 2]. Starting from what seems to be the first published model of hypercomputation, which is called Turing's oracle machine [3]; the formulations of models and algorithms of hypercomputation have been applied a wide spectrum of underlying theories [1, 4, 5]. It is precisely due to the existence of Turing's oracle machines that J Copeland and D Proudfoot introduced

the term 'hypercomputation' in 1999 [6] to replace wrong expressions such as 'super-Turing computation', 'computing beyond Turing's limit', 'breaking the Turing barrier' and similar.

Recently, Tien D Kieu has proposed a quantum algorithm to solve the TM incomputable[1] problem named Hilbert's tenth problem, using as physical referent the well-known simple harmonic oscillator (SHO), which by effect of the second quantization, it is a realization of dynamical algebra Weyl–Heisenberg denoted by $\mathfrak{g}_{\text{W–H}}$ [8–13]. From the algebraic analysis of Kieu's hypercomputational quantum algorithm (KHQA), we have identified the underlying properties of the $\mathfrak{g}_{\text{W–H}}$ algebra which are necessary (but not sufficient) to guarantee KHQA works. Such properties are that the dynamical algebra admits infinite-dimensional irreducible representations with naturally associated coherent states.

The importance of KHQA in the field of hypercomputation, at the same tenor as the importance of hypercomputation within the domain of computer science, cannot be underestimated. This algorithm is a plausible candidate for a practical implementation of the hypercomputation, maybe within the scope of quantum optics. The adaptation of KHQA to new physical referents different than the harmonic oscillator opens the possibility of analyse news viable alternatives for its practical implementation beyond quantum optics, maybe using quasi-particles of condensed matter systems.

In this work, we present an algebraic adaptation of KHQA, that is to say, we present a hypercomputation model à la Kieu, based on the selection of a dynamical algebra which is different than the $\mathfrak{g}_{\text{W–H}}$ algebra. We have selected the Lie algebra $\mathfrak{su}(1,1)$, because this algebra possesses the necessary characteristics to realize the hypercomputation and also because such algebra has been identified as the dynamical algebra associated with many relatively simple quantum systems.

More concretely, the $\mathfrak{su}(1,1)$ algebra possesses four kinds of infinite-dimensional unitary irreducible representations (UIR): the positive discrete series, the negative discrete series, the principal series and the complementary series [14]. In this work, we use only the positive discrete series. From the other side, the $\mathfrak{su}(1,1)$ algebra admits different kinds of coherent states such as Barut–Girardello, Perelomov, nonlinear and minimum uncertain [15, 16]. In addition to all of these, the $\mathfrak{su}(1,1)$ algebra admits different kinds of realizations. Within the field of quantum optics, we have realizations in systems with one, two and four photon modes [15, 16], or with systems such as the density-dependent Holstein–Primakoff [15]. Within the domain of condensed matter, we have realizations in the following quantum potentials: infinite square well, the Pöschl–Teller potentials [17] and Calogero–Sutherland model [16]. Other realizations from the $\mathfrak{su}(1,1)$ algebra arise from the mathematical physics in relation with the recursive properties of the special functions, namely Laguerre oscillators [18–20], Legendre and Chebyshev oscillators [19], Meixner Oscillators [14, 21] and so on.

The present paper is realized in the following way. In section 2 we introduce KHQA in such a way that the algebraic issues have been empathized and we make explicit the hypercomputational characteristics of the $\mathfrak{g}_{\text{W–H}}$ algebra. In section 3, based on the analysis of such algebraic characteristics, we show the general structure and the mathematical properties of our adaptation of KHQA using the $\mathfrak{su}(1,1)$ algebra. In section 4 we note that the infinite cylindrical well and a modified cylindrical well also admit a realization of the $\mathfrak{su}(1,1)$ algebra. Moreover, based on the adaptation of KHQA that we have realized using the infinite square well [22, 23], we show new adaptations of KHQA for some of the physical referents previously listed. Finally, we present some conclusions.

[1] We follow S B Cooper and P Odifreddi, and we adopt the terminology Turing's 'computable' in place of Kleene's 'recursive' (see footnote 1 in [7]).

2. Kieu's hypercomputational quantum algorithm

Based on the SHO and its associated dynamical algebra $\mathfrak{g}_{\text{W–H}}$, Kieu has proposed a possible algorithm for the solution of Hilbert's tenth problem by the use of three strategies: (i) codification of the instance of the Hilbert's tenth problem to solve, (ii) the utilization of a non-standard version of quantum computation, and (iii) the establishment of a halting criterion. The strategy (i) has a background in the occupation-number operator associated with the $\mathfrak{g}_{\text{W–H}}$ algebra. The strategy (ii) is based on the adiabatic quantum computation [24, 25] applied to unbounded Hamiltonians, that is to say, this strategy constitutes an application of the quantum adiabatic theorem for the case of unbounded operators [26, 27]. The adiabatic initialization is obtained with the aid of the coherent states and the ladder operators which are associated with the dynamical algebra $\mathfrak{g}_{\text{W–H}}$. The strategy (iii) demands a property to the initial state of the adiabatic evolution. Such property is based on the probability distribution associated with the coherent states corresponding to the $\mathfrak{g}_{\text{W–H}}$ algebra. We now present in detail, every one of the strategies previously enunciated, in such way that the possible algebraic generalizations can arise easily.

2.1. Mathematical background

The mathematical background underlying KHQA is shown by equations (1) and corresponds to the mathematical formalism of the SHO within the formulation of the second quantization. At (1*a*) we introduce the Fock occupation-number states denoted by $\mathfrak{F}^{\text{SHO}}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of non-negative integers. At (1*b*) the annihilation and creation operators a and $a^\dagger$ are introduced. The commutation relations between the ladder operators are presented in (1*c*). At (1*d*) the spectral equation for the SHO is shown in terms of the Hamiltonian H^{SHO} and of the energy levels E_n^{SHO}. At (1*e*) the Hamiltonian H^{SHO} is given in terms of the ladder operators. At (1*f*) is presented the definition of the occupation-number operator N^{SHO} whose eigenvalues are denoted by n and which will be crucial for what follows. Equation (1*g*) gives the definition and the explicit form of the coherent states denoted by $|\alpha\rangle^{\text{SHO}}$. Finally (1*h*) shows the Poisson form of the probability density for the random variable n corresponding to the coherent states (1*g*).

$$\mathfrak{F}^{\text{SHO}} = \{|n\rangle \mid n \in \mathbb{N}\}, \tag{1a}$$

$$a|0\rangle = 0, \qquad a|n\rangle = \sqrt{n}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \tag{1b}$$

$$[a, a^\dagger] = \mathbb{1} \tag{1c}$$

$$H^{\text{SHO}}|n\rangle = E_n^{\text{SHO}}|n\rangle, \tag{1d}$$

$$H^{\text{SHO}} = \hbar\omega^{\text{SHO}}(a^\dagger a + 1/2), \tag{1e}$$

$$N^{\text{SHO}} = a^\dagger a, \qquad N^{\text{SHO}}|n\rangle = n|n\rangle, \tag{1f}$$

$$\begin{aligned} a|\alpha\rangle^{\text{SHO}} &= \alpha|\alpha\rangle^{\text{SHO}} \\ &= \mathrm{e}^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle, \qquad \text{where } \alpha \in \mathbb{C}, \end{aligned} \tag{1g}$$

$$P_n^{\text{SHO}}(\alpha) = \mathrm{e}^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}. \tag{1h}$$

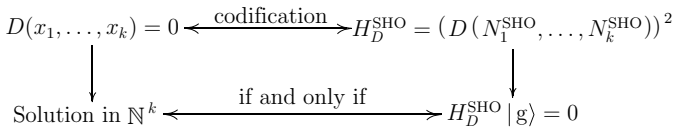

Figure 1. Kieu's codification.

2.2. First strategy: Hilbert's tenth problem and its codification

A Diophantine equation is an equation of the form

$$D(x_1,\ldots,x_k)=0, \tag{2}$$

where D is a polynomial with integer coefficients. In 1900, David Hilbert presented his famous list of 23 problems. From this list we extract problem number 10. In present terminology, Hilbert's tenth problem may be paraphrased as

> Given a Diophantine equation with any number of unknowns: To devise a process according to which it can be determined by a finite number of operations whether the equation has non-negative integer solutions.

From the concluding results obtained by Matiyasevich, Davis, Robinson and Putnam, we know actually that, in the general case, this problem is algorithmically insolvable, or more precisely, it is TM incomputable [28]. Actually, the possible hypercomputability of Kieu's algorithm is due to the fact that this algorithm solves Hilbert's tenth problem.

With the mathematical background that was presented at (1), Kieu proposes the codification of Hilbert's tenth problem which is presented in figure 1. This figure illustrates that a Diophantine equation of the kind (2) is codified on a Hamiltonian denoted by H_{D} which results from the substitution of every unknown in (2) by the occupation-number operator defined in (1*f*). In this way, the problem to determine if (2) has solutions within the non-negative integers is equivalent to the problem to determine if the energy associated with the fundamental state denoted by $|g\rangle$, of the Hamiltonian H_{D} is zero.

2.3. Second strategy: quantum adiabatic computation

Due to the codification showed in figure 1, it is necessary to use a strategy of quantum computation which is different from the standard quantum computation (based on sequences of unitary quantum logic gates that process qubits) [29]. Using the words of Kieu, we present the strategy of quantum computation in the following form [12, p 7]:

> In general, it is much more difficult to construct a specific state for a quantum mechanical system than to control the physical process (that is, to create a corresponding Hamiltonian) to which the system is subject. One systematic method to obtain the ground state of a Hamiltonian is to exploit the quantum adiabatic theorem to reach the desired state through some adiabatic evolution which starts from a readily constructible ground state of some other Hamiltonian. This is the idea of quantum adiabatic computation (QAC) [24] ...
>
> In QAC, we encode the solution of our problem to the ground state of some specific Hamiltonian. As it is easier to implement controlled dynamical processes than to obtain the ground state, we start the computation with the system prepared in a different but readily obtainable ground state of some other Hamiltonian. This initial Hamiltonian is then slowly extrapolated into the Hamiltonian whose ground state is the desired one. The adiabatic theorem of quantum mechanics (QAT) [26] stipulates

that if the extrapolation rate is sufficiently slow compared to some intrinsic scale, the initial state will evolve into the desired ground state with a high probability ... Measurements then take place finally on the system in order to identify the ground state, from which the solution to our problem emerges ...

Now, to carry out a QAC for a given Diophantine equation (2), we prepare our quantum-mechanical system in the readily constructible initial ground state

$$|g_I\rangle^{\mathrm{SHO}} = \bigotimes_{i=1}^{k} |\alpha_i\rangle^{\mathrm{SHO}}, \tag{3}$$

of a *universal* (that is, independent of the given Diophantine equation) initial Hamiltonian H_I, with some complex numbers α's,

$$H_I^{\mathrm{SHO}} = \sum_{i=1}^{k} \left(a_i^{\dagger} - \alpha_i^*\right)(a_i - \alpha_i). \tag{4}$$

This is just the Hamiltonian for shifted simple harmonic oscillators whose ground state is the well-known coherent state in quantum optics. We then subject the system to the time-dependent Hamiltonian H_A^{SHO}, which linearly extrapolates the initial Hamiltonian H_I to the final Hamiltonian H_D in a time interval T,

$$H_A^{\mathrm{SHO}}(t) = \left(1 - \frac{t}{T}\right) H_I^{\mathrm{SHO}} + \frac{t}{T} H_D^{\mathrm{SHO}}. \tag{5}$$

2.4. The algorithm

Based on the two mentioned strategies and the strategy that will be presented in the following section, given a Diophantine equation with k unknowns of type (2), Kieu provides the following quantum algorithm to decide whether this equation has a non-negative integer solution or not [10, 12].

(i) Construct a physical process in which a system initially starts with a direct product of k coherent states

$$|\psi(0)\rangle = |g_I\rangle^{\mathrm{SHO}},$$

and in which the system is subject to a time-dependent Hamiltonian $H_A^{\mathrm{SHO}}(t)$ of (5) over the time interval $[0, T]$, for some time T.

(ii) Measure through the time-dependent Schrödinger equation

$$\mathrm{i}\partial_t |\psi(t)\rangle = H_A^{\mathrm{SHO}}(t)|\psi(t)\rangle, \qquad \text{for} \quad t \in [0, T]$$

the maximum probability to find the system in a particular occupation-number state at the chosen time T,

$$\begin{aligned} P(T) &= \max_{|\{n\}\rangle} |\langle\psi(T) \mid \{n\}\rangle|^2 \\ &= |\langle\psi(T) \mid \{n\}^0\rangle|^2, \end{aligned}$$

where $|\{n\}\rangle = \bigotimes_{i=1}^{k} |n_i\rangle$, and $|\{n\}^0\rangle$ is the maximum-probability number state with a particular k-tuple $\left(n_1^0, \ldots, n_k^0\right)$.

(iii) If $P(T) \leqslant 1/2$, increase T and repeat all the steps above.

(iv) If

$$P(T) > 1/2 \tag{6}$$

then $|\{n\}^0\rangle$ is the ground state of H_D^{SHO} (assuming no degeneracy) and we can terminate

the algorithm and deduce a conclusion from the fact that $H_D^{\mathrm{SHO}}|\{n\}^0\rangle = 0$ iff equation (2) has a non-negative integer solution.

2.5. *Third strategy: the halting criterion*

One of the most common misunderstandings about KHQA is linked to the halting criterion of the algorithm. Some authors claim that the QAT only establishes the existence of a time of execution of the algorithm which is finite but unknown and then there is not a verifiable halting criterion of the algorithm. From the very early versions of the algorithm, Kieu has been alert to this situation and he has proposed the following halting criterion [12, p 9–11]:

> Nevertheless, it is important to note that QAT is not constructive, as with most theorems involving limiting processes. It only tells us that for 'sufficiently large' T the system is 'mostly' in the instantaneous eigenstate. But the theorem tells us nothing quantitatively about the degrees of being 'sufficiently large' or 'mostly' ...
>
> In other words, QAT can only guarantee that the ground state is achievable in a finite time interval but cannot specify what that interval should be. That is, it cannot by itself give us any indication when the ground state has been obtained so that the algorithm can then be terminated at that point. For that, we need another criterion ...
>
> The crucial step of any quantum adiabatic algorithm is the identification of the ground state of the final Hamiltonian, H_D. In our case we do not in advance know in general how long is sufficiently long (the theorem offers no direct help here); all we can confidently know is that for each Diophantine equation and each set of α_i's there is a *finite* evolution time after which the adiabaticity condition is satisfied. We thus have to find another criterion to identify the ground state.
>
> The identification criterion we have found can be stated as: the ground state of H_D^{SHO} is the Fock state $|\{n\}^0\rangle$ measurably obtained with a probability of more than $1/2$ after the evolution for some time T of the initial ground state $|\mathrm{g}_I\rangle^{\mathrm{SHO}}$ according to the Hamiltonian (5):
>
> $$|\{n\}^0\rangle \text{ is the ground state of } H_D^{\mathrm{SHO}} \text{ if } |\langle\psi(T)\mid\{n\}^0\rangle|^2 > 1/2, \qquad \text{for some } T,$$
>
> provided the initial ground state $|\mathrm{g}_I\rangle^{\mathrm{SHO}}$ of H_I^{SHO} does not have any dominant component in the occupation-number eigenstates $|\{n\}\rangle$ of H_D^{SHO},
>
> $$|^{\mathrm{SHO}}\langle \mathrm{g}_I|\{n\}\rangle|^2 \leqslant 1/2, \qquad \forall\,\{n\}; \tag{7}$$
>
> and provided that for $0 < t < T$,
>
> $$\langle \mathrm{e}(t)|H_D^{\mathrm{SHO}} - H_I^{\mathrm{SHO}}|\mathrm{g}(t)\rangle \neq 0, \tag{8}$$
>
> where $|\mathrm{g}(t)\rangle$ and $|\mathrm{e}(t)\rangle$ are, respectively, the instantaneous ground state and the first excited state of H_A^{SHO} at the time t.[2]

2.6. *Crucial properties of the* $\mathfrak{g}_{\mathrm{W\text{–}H}}$ *algebra for KHQA*

According to (1), the dynamical algebra realized by SHO is the Lie algebra $\mathfrak{g}_{\mathrm{W\text{–}H}}$, whose generators are the operators a, $a^\dagger$ and $\mathbb{1}$. The $\mathfrak{g}_{\mathrm{W\text{–}H}}$ algebra admits a infinite-dimensional UIR,

[2] Criterion (8) was added recently by Kieu [12, 30] to correct the finite-dimensional counter-examples pointed out by Smith [31]. On the other hand, there was an open problem in relation to infinite-dimensional case. In a personal communication, Kieu told us that he has found a mathematical proof that halting criterion (6) is a good identification for the ground state in this case [32].

which is established by the action of its generators over the space $\mathfrak{F}^{\mathrm{SHO}}$ and which is given by (1*b*). From this representation, the occupation-number operator N^{SHO} is obtained whose spectrum coincides with the non-negative integers $\mathbb{N}$ as is shown by (1*f*). This spectrum is precisely the searching space of the solution, associated with every one of the variables of (2) and justifies the strategy of codification which is shown in figure 1.

From the other side, the adiabatic initialization for KHQA, which is represented by (3) and (4), comes from the $\mathfrak{g}_{\mathrm{W\text{–}H}}$ algebra. The initial state $|g_I\rangle^{\mathrm{SHO}}$ is the direct product of k coherent states of the form (1*g*), and the initial Hamiltonian denoted by H_I^{SHO} is constructed starting from the ladder operators $a^\dagger$ and a of the $\mathfrak{g}_{\mathrm{W\text{–}H}}$ algebra. Besides this, the identification of the ground state of H_D^{SHO} that assumes the role of halting criterion for the algorithm according to (6) is supported by condition (7), which is satisfied by the probability density P_n^{SHO} of (1*h*). In concrete, the chosen of the coherent state with the form (3) as the initial ground state entails condition (7), since for any $\alpha \neq 0$, and $\forall n > 0$

$$|\langle \alpha \mid n\rangle|^2 = P_n^{\mathrm{SHO}}(\alpha) < 1/2.$$

3. Hypercomputational quantum algorithm based on the algebra $\mathfrak{su}(1,1)$

From the algebraic point of view, the peculiarities of the algebra $\mathfrak{g}_{\mathrm{W\text{–}H}}$, which are required by KHQA, start from the fact that this algebra admits an infinite-dimensional UIR that operates over the Fock space and its corresponding coherent states. Based on such infinite-dimensional UIR is possible to establish the needed ladder operators that let the construction of a number operator and the associated coherent states, the basic algebraic ingredients of KHQA.

Due to the fact that the $\mathfrak{g}_{\mathrm{W\text{–}H}}$ algebra is not the only dynamical algebra that satisfies the needed algebraic conditions, the problem then arises of the adaptation of KHQA to other dynamical algebras and then to other physical systems.

We present in this section the adaptation of KHQA to the case of the $\mathfrak{su}(1,1)$ algebra. Such algebra is chosen because this algebra is the dynamical algebra associated with many well-known physical systems.

The algebra $\mathfrak{su}(1,1)$ is defined by the commutation relations

$$[K_0, K_1] = \mathrm{i}K_2, \qquad [K_0, K_2] = -\mathrm{i}K_1, \qquad [K_1, K_2] = -\mathrm{i}K_0,$$

or by the commutation relations

$$[K_0, K_\pm] = \pm K_\pm, \qquad [K_+, K_-] = -2K_0, \qquad \text{where} \quad K_\pm \equiv (K_1 \pm \mathrm{i}K_2). \tag{9}$$

In contrast with the $\mathfrak{g}_{\mathrm{W\text{–}H}}$ algebra, the algebra $\mathfrak{su}(1,1)$ admits different kinds of coherent states in addition to various kinds of representations. Here we use the named positive discrete representation, which is defined as [15, 16]

$$\begin{aligned} K_-|k,n\rangle &= \sqrt{n(2k+n-1)}|k,n-1\rangle, \\ K_+|k,n\rangle &= \sqrt{(n+1)(2k+n)}|k,n+1\rangle, \\ K_0|k,n\rangle &= (n+k)|k,n\rangle, \end{aligned} \tag{10}$$

where $|k,n\rangle (n \in \mathbb{N})$ is the normalized basis and $k \in \left\{\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\right\}$ is the Bargmann index labeling the UIR[3].

We introduce the number operator $N^{\mathfrak{su}(1,1)}$ by

$$N^{\mathfrak{su}(1,1)} = K_0 - k, \qquad N^{\mathfrak{su}(1,1)}|k,n\rangle = n|k,n\rangle, \tag{11}$$

[3] Following Antoine *et al*, we recall that $k \in \left\{\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\right\}$ for the discrete series *stricto senso* UIRs of $\mathfrak{su}(1,1)$, and $k \in [1/2, +\infty)$ for the extension to the universal covering of the group $SU(1,1)$ [17]. Whatever the fact, we will speak about discrete series UIRs for both cases.

and the well-known coherent states of $\mathfrak{su}(1,1)$ are the denominated Barut–Girardello coherent states (BGCS). The BGCS are defined as the eigenstates of the lowering operator K_-

$$K_-|k,\alpha\rangle_{\mathrm{BG}} = \alpha|k,\alpha\rangle_{\mathrm{BG}}, \tag{12}$$

and these can be expressed as [15]

$$|k,\alpha\rangle_{\mathrm{BG}} = \sqrt{\frac{|\alpha|^{2k-1}}{I_{2k-1}(2|\alpha|)}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!\Gamma(n+2k)}}|k,n\rangle, \tag{13}$$

where $I_\nu(x)$ is a modified Bessel function of the first kind.

All this is well known. Now, we introduce a generalizations of equations (9)–(13), which we will use for that which follows. Assuming the existence of a quantum system S, whose dynamical algebra is $\mathfrak{su}(1,1)$, equations (14) gives the adaptation of equations (1) for the case of $\mathfrak{su}(1,1)$. Equation (14*a*) defines the Fock space of quantum states, which is denoted by $\mathfrak{F}^{\mathrm{S}}$ corresponding to a quantum system denoted by S. Equation (14*b*) presents the commutation relations that defines the $\mathfrak{su}(1,1)$ algebra, where the generators K_+^{S} and K_-^{S} correspond respectively to the creation and destruction operators of $\mathfrak{su}(1,1)$. Equation (14*c*) shows the action of the infinite-dimensional UIR of $\mathfrak{su}(1,1)$ over the space $\mathfrak{F}^{\mathrm{S}}$, with characteristic function f^{S}, which is assumed to be a quadratic function of n. Expression (14*d*) shows the equation of the energy spectrum for the quantum system, and (14*e*) gives the factorized form of the Hamiltonian H^{S} in terms of the ladder operators. Equation (14*f*) defines the number operator associated with the system S and its corresponding action over the space $\mathfrak{F}^{\mathrm{S}}$. Equation (14*g*) is a definition of a nonlinear coherent state denoted by $|z\rangle$, which is a generalization of the more standard linear coherent states of the Barut–Girardello and Klauder–Perelomov [15, 33] kinds; and (14*h*) presents the form of such nonlinear coherent states (see the appendix). Equation (14*i*) shows the probability density for the random variable n, which is associated with the generalized coherent states (14*h*).

$$\mathfrak{F}^{\mathrm{S}} = \{|n\rangle \mid n \in \mathbb{N}\}, \tag{14a}$$

$$\left[K_-^{\mathrm{S}}, K_+^{\mathrm{S}}\right] = K_3^{\mathrm{S}}, \qquad \left[K_-^{\mathrm{S}}, K_3^{\mathrm{S}}\right] = 2K_-^{\mathrm{S}}, \qquad \left[K_+^{\mathrm{S}}, K_3^{\mathrm{S}}\right] = -2K_+^{\mathrm{S}}, \tag{14b}$$

$$\begin{aligned} &K_-^{\mathrm{S}}|0\rangle = 0, \qquad K_-^{\mathrm{S}}|n\rangle = \sqrt{f^{\mathrm{S}}(n)}|n-1\rangle, \\ &K_+^{\mathrm{S}}|n\rangle = \sqrt{f^{\mathrm{S}}(n+1)}|n+1\rangle, \qquad K_3^{\mathrm{S}}|n\rangle = (f^{\mathrm{S}}(n+1) - f^{\mathrm{S}}(n))|n\rangle = g^{\mathrm{S}}(n)|n\rangle, \end{aligned} \tag{14c}$$

$$H^{\mathrm{S}}|n\rangle = E_n^{\mathrm{S}}|n\rangle, \tag{14d}$$

$$H^{\mathrm{S}} = \hbar\omega\left(K_+^{\mathrm{S}}K_-^{\mathrm{S}}\right), \qquad \text{with} \quad K_+^{\mathrm{S}}K_-^{\mathrm{S}}|n\rangle = f^{\mathrm{S}}(n)|n\rangle, \tag{14e}$$

$$N^{\mathrm{S}} = (f^{\mathrm{S}}(H^{\mathrm{S}}))^{-1} = \left(g^{\mathrm{S}}\left(K_3^{\mathrm{S}}\right)\right)^{-1}, \qquad N^{\mathrm{S}}|n\rangle = n|n\rangle, \tag{14f}$$

$$h^{\mathrm{S}}(N^{\mathrm{S}})K_-^{\mathrm{S}}|z\rangle^{\mathrm{S}} = z|z\rangle^{\mathrm{S}}, \qquad z \in \mathbb{C}, \tag{14g}$$

$$|z\rangle^{\mathrm{S}} = \left(\sum_{m=0}^{\infty}\frac{|z|^{2m}}{\left(\prod_{j=0}^{m-1}h^{\mathrm{S}}(j)\right)^2(f^{\mathrm{S}}(m)!)}\right)^{-1/2}\sum_{n=0}^{\infty}\frac{z^n}{\left(\prod_{j=0}^{n-1}h^{\mathrm{S}}(j)\right)\left(\sqrt{f^{\mathrm{S}}(n)!}\right)}, \tag{14h}$$

$$P_n^{\mathrm{S}}(z) = \left(\sum_{m=0}^{\infty}\frac{|z|^{2m}}{\left(\prod_{j=0}^{m-1}h^{\mathrm{S}}(j)\right)^2(f^{\mathrm{S}}(m)!)}\right)^{-1}\frac{|z|^{2n}}{\left(\prod_{j=0}^{n-1}h^{\mathrm{S}}(j)\right)^2(f^{\mathrm{S}}(n)!)}. \tag{14i}$$

Starting from equations (14), the annunciated adaptation of KHQA for the case of the $\mathfrak{su}(1,1)$ algebra is completely direct. In the strategy of codification, which was represented

by figure 1, it is necessary to replace the Hamiltonian H_D^{SHO} with a new Hamiltonian denoted by H_D^{S} which is constructed with the number operators defined at (14*f*), by means of

$$H_D^{\mathrm{S}} = \left(D\left(N_1^{\mathrm{S}}, \ldots, N_k^{\mathrm{S}}\right)\right)^2. \tag{15}$$

The adiabatic initialization is obtained from the coherent states (14*h*) and from (14*g*), and is given by

$$|g_I\rangle^{\mathrm{S}} = \bigotimes_{i=1}^{k} |z_i\rangle^{\mathrm{S}}, \tag{16}$$

$$H_I^{\mathrm{S}} = \sum_{i=1}^{k} \left(K_{+_i}^{\mathrm{S}} h^{\mathrm{S}}(N^{\mathrm{S}}) - z_i^*\right)\left(h^{\mathrm{S}}(N^{\mathrm{S}}) K_{-_i}^{\mathrm{S}} - z_i\right). \tag{17}$$

From all this, we obtain the Hamiltonian denoted by H_A^{S}, which is the generator of the adiabatic evolution and which is of the form

$$H_A^{\mathrm{S}}(t) = \left(1 - \frac{t}{T}\right) H_I^{\mathrm{S}} + \frac{t}{T} H_D^{\mathrm{S}}. \tag{18}$$

Finally, to satisfy the halting criterion (6), we chose a value for the parameter $z \in \mathbb{C}$, which according to (14*i*) satisfies the condition in (7), it is to say we chose a value of z such that

$$P_n^{\mathrm{S}}(z) < 1/2.$$

All that is the abstract generalization or extension of Kieu's original hypercomputational quantum algorithm. In the following section, we will present some concrete physical referents on which to realize the implementation of the abstract algorithm previously presented.

4. Adaptation of KHQA over some concrete physical referents

In this section, some concrete quantum systems are presented as possible physical referents on which to try to implement the adaptation of KHQA with the $\mathfrak{su}(1,1)$ algebra, which was presented in the past section. The mathematical adaptation expressed by equations (14) depends on the particular forms of the characteristic functions f^{S} and h^{S} associated with the physical system S. Then for every one of the considered physical systems, we establish that the corresponding dynamical algebra is precisely $\mathfrak{su}(1,1)$ and we determine the particular forms of f^{S} and h^{S}. The physical systems that are considered here are the infinite square well, the Pöschl–Teller potentials, the infinite cylindrical well, a perturbed cylindrical well, the density-dependent Holstein–Primakoff system of quantum optics and the Laguerre oscillator. Other systems of quantum optics such as two-mode realization, amplitude-squared realization and four-mode system admit also infinite-dimensional representations but such representations are reducible and with such kind of representations it is more difficult to adapt KHQA.

4.1. The infinite square well

The adaptation of the KHQA for the case of the infinite square well (ISW) was realized by the present authors within a previous work [22, 23]. In the present work, we again establish that the ISW satisfies the mathematical structure given by (14), for particular forms of f^{ISW} and h^{ISW} from which it is possible to construct the constitutive elements of the basic algebraic anatomy of the KHQA.

For a particle with mass m that is trapped inside the infinite square well $0 \leqslant x \leqslant \pi l$, the Fock space associated $\mathfrak{F}^{\mathrm{ISW}}$, the Hamiltonian operator H^{ISW}, the eigenvalue equation and the boundary conditions are [17]

$$\mathfrak{F}^{\mathrm{ISW}} = \{|n\rangle \mid n \in \mathbb{N}\}, \qquad H^{\mathrm{ISW}} = i^2 \frac{\hbar^2}{2m} \frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\hbar^2}{2ml^2}, \tag{19}$$

$$H^{\mathrm{ISW}} \psi^{\mathrm{ISW}} = E^{\mathrm{ISW}} \psi^{\mathrm{ISW}}, \tag{20}$$

$$\psi(x) = 0, \qquad x \geqslant \pi l \qquad \text{and} \qquad x \leqslant 0. \tag{21}$$

Replacing (19) on (20) together with the boundary conditions (21), we obtain

$$\begin{aligned} &\psi_n^{\mathrm{ISW}}(x) = \sqrt{\frac{2}{\pi l}} \sin(x+1)\frac{x}{l} \equiv \langle x \mid n\rangle, \\ &E_n^{\mathrm{ISW}} = \hbar\omega^{\mathrm{ISW}} e_n^{\mathrm{ISW}}, \qquad \text{where} \quad \omega^{\mathrm{ISW}} = \frac{\hbar^2}{2ml^2} \quad \text{and} \quad e_n^{\mathrm{ISW}} = n(n+2), \quad n \in \mathbb{N}, \end{aligned} \tag{22}$$

and where the action of H^{ISW} over the space $\mathfrak{F}^{\mathrm{ISW}}$ is given by

$$H^{\mathrm{ISW}}|n\rangle = E_n^{\mathrm{ISW}}|n\rangle.$$

Due to the spectral structure of the ISW, the dynamical algebra associated with it is the Lie algebra $\mathfrak{su}(1,1)$ [17] whose generators denoted by K_+^{ISW}, K_-^{ISW} and K_3^{ISW} satisfy the commutation relations of (14*b*). Based on (22), the algebra $\mathfrak{su}(1,1)$ admits an infinite-dimensional UIR over the space $\mathfrak{F}^{\mathrm{ISW}}$, which is given by

$$\begin{aligned} K_-^{\mathrm{ISW}}|0\rangle &= 0, \\ K_-^{\mathrm{ISW}}|n\rangle &= \sqrt{e_n^{\mathrm{ISW}}}|n\rangle = \sqrt{n(n+2)}|n-1\rangle, \\ K_+^{\mathrm{ISW}}|n\rangle &= \sqrt{e_{n+1}^{\mathrm{ISW}}}|n\rangle = \sqrt{(n+1)(n+3)}|n+1\rangle, \\ K_3^{\mathrm{ISW}}|n\rangle &= \left(e_{n+1}^{\mathrm{ISW}} - e_n^{\mathrm{ISW}}\right)|n\rangle = (2n+3)|n\rangle. \end{aligned}$$

Based on this representation of the algebra $\mathfrak{su}(1,1)$, the Hamiltonian (19) is rewritten as

$$H^{\mathrm{ISW}} = \hbar\omega\left(K_+^{\mathrm{ISW}} K_-^{\mathrm{ISW}}\right), \qquad H^{\mathrm{ISW}}|n\rangle = E_n^{\mathrm{ISW}}|n\rangle,$$

and a new number operator N^{ISW} is given by

$$N^{\mathrm{ISW}} = (1/2)\left(K_3^{\mathrm{ISW}} - 3\right), \qquad N^{\mathrm{ISW}}|n\rangle = n|n\rangle.$$

Due to the associated dynamical algebra, the BGCS $|z\rangle^{\mathrm{ISW}}$, $z \in \mathbb{C}$, for the ISW are given by [34]

$$K_-^{\mathrm{ISW}}|z\rangle^{\mathrm{ISW}} = z|z\rangle^{\mathrm{ISW}}, \qquad \text{where} \quad |z\rangle^{\mathrm{ISW}} = \frac{|z|}{\sqrt{I_2(2\,|z|)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!(n+2)!}}|n\rangle, \tag{23}$$

where $I_v(x)$ is the modified Bessel function of the first kind. The corresponding probability density for the random variable n that results from (23) is

$$P_n^{\mathrm{ISW}}(z) = \frac{|z|^2}{I_2(2|z|)} \frac{|z|^{2n}}{n!(2+n)!}.$$

We have established then that the ISW satisfies the algebraic structure of (14) where the characteristic functions are of the forms

$$f^{\mathrm{ISW}}(n) = e_n^{\mathrm{ISW}} = n(n+2), \qquad h^{\mathrm{ISW}}(N^{\mathrm{ISW}}) = \mathbb{1},$$

and we can in consequence to rewrite in terms of the ISW, all the elements of the KHQA, which are given by equations (15)–(18) and to obtain in this way, an adaptation of the KHQA for the ISW, on where the halting criterion (6) is satisfied according to condition (7) with every one value of $z \in \mathbb{Z}$ that verifies that

$$P_0^{\mathrm{ISW}}(z) = \frac{|z|^2}{2I_2(2|z|)} < 1/2,$$

given that $P_n^{\mathrm{ISW}}(z) \leqslant P_0^{\mathrm{ISW}}(z), \forall\, n$.

4.2. The infinite cylindrical well and the perturbed cylindrical well

With the aim to adapt the KHQA over the infinite cylindrical well (ICW) or over a perturbed cylindrical well (PCW), initially we establish that these physical referents have as dynamical algebra just the $\mathfrak{su}(1,1)$ algebra. We follow the work that was realized by Antoine *et al* [17], in this way that we obtain the particular forms of the functions $f^{\mathrm{ICW/PCW}}$ and $h^{\mathrm{ICW/PCW}}$ that are required.

In concrete, for a particle with mass m which is trapped inside the infinite cylindrical well of radius R, the associated Fock space denoted by $\mathfrak{F}^{\mathrm{ICW}}$ and the corresponding Hamiltonian operator H^{ICW} are given by

$$\mathfrak{F}^{\mathrm{ISW}} = \{|n\rangle \mid n \in \mathbb{N}\}, \qquad H^{\mathrm{ICW}} = -\frac{\hbar^2}{2m}\nabla^2 + U^{\mathrm{ICW}}, \tag{24}$$

where U^{ICW} is a constant that will be obtained posteriorly and the bi-dimensional Laplacian operator is written on cylindrical coordinates. The spectral equation for Hamiltonian H^{ICW} is

$$H^{\mathrm{ICW}}\Psi^{\mathrm{ICW}} = E^{\mathrm{ICW}}\Psi^{\mathrm{ICW}}. \tag{25}$$

Now the substitution of (24) on (25) gives the following partial differential equation whose solution determines the spectrum of H^{ICW}

$$-\frac{\hbar^2}{2m}\nabla^2\Psi^{\mathrm{ICW}} + U^{\mathrm{ICW}}\Psi^{\mathrm{ICW}} = E^{\mathrm{ICW}}\Psi^{\mathrm{ICW}}. \tag{26}$$

Using cylindrical coordinates and axial symmetry, (26) is reduced to

$$\frac{\partial^2}{\partial r}\Psi^{\mathrm{ICW}}(r) + \frac{1}{r}\frac{\partial}{\partial r}\Psi^{\mathrm{ICW}}(r) + \frac{2m}{\hbar^2}(E^{\mathrm{ICW}} - U^{\mathrm{ICW}})\Psi^{\mathrm{ICW}}(r) = 0. \tag{27}$$

The condition of trapping for the particle within the interior of the ICW is introduced using the boundary condition

$$\Psi^{\mathrm{ICW}}(R) = 0. \tag{28}$$

The solution of (27) with the condition of wavefunction finite at $r = 0$ is given by

$$\Psi^{\mathrm{ICW}}(r) = C\,J_0\left(\sqrt{\frac{2m(E^{\mathrm{ICW}} - U^{\mathrm{ICW}})}{\hbar^2}}\,r\right), \tag{29}$$

where C is a constant. Now using (29) and the boundary condition (28), the energy spectrum is obtained as

$$E_n^{\mathrm{ICW}} = U^{\mathrm{ICW}} + \frac{\hbar^2}{2mR^2}\alpha_n^2, \tag{30}$$

where $n \in \mathbb{N}$ and the α_n's are the zeros of the Bessel function $J_0(x)$. Using the empirical formula of interpolation for α_n

$$\alpha_n = 3.115n + 2.405, \tag{31}$$

the substitution of (31) on (30) gives

$$E_n^{\mathrm{ICW}} = U^{\mathrm{ICW}} + \frac{2.89\hbar^2}{mR^2} + \frac{4.85\hbar^2}{mR^2} n(n+1.54)$$
$$= \hbar\omega^{\mathrm{ICW}} e_n^{\mathrm{ICW}}, \tag{32}$$

where $U^{\mathrm{ICW}} = -2.89\hbar^2/mR^2$, $\omega^{\mathrm{PCW}} = 4.85\hbar/mR^2$ and $e_n^{\mathrm{ICW}} = n(n+1.54)$, $n \in \mathbb{N}$.

The normalized wavefunction is given by

$$\Psi_n^{\mathrm{ICW}}(r) = \frac{1}{R\sqrt{\pi}} \frac{J_0\left(\frac{\alpha_n}{R} r\right)}{J_1(\alpha_n)} \equiv \langle r \mid n \rangle,$$

and the action of H^{ICW} over the space $\mathfrak{F}^{\mathrm{ISW}}$ being

$$H^{\mathrm{ICW}}|n\rangle = \hbar\omega^{\mathrm{ICW}} e_n^{\mathrm{ICW}}|n\rangle.$$

From the other side, for the case of the PCW, we consider a quantum particle that it is confined to the interior of an infinite long cylinder of finite radius R but now the interior of the cylinder has a potential of the form

$$V^{\mathrm{PCW}}(r) = W^{\mathrm{PCW}} + \frac{U^{\mathrm{PCW}}}{r^2}, \tag{33}$$

where U^{PCW} and W^{PCW} are constants that we can determine ulteriorly, and we assume that both the wall of cylinder and the axis of the cylinder always obstruct that the particle resides on them, because both the wall and the axis are maintained to infinite potential.

For a particle with mass m trapped inside the PCW of radius R, the associated Fock space denoted by $\mathfrak{F}^{\mathrm{PCW}}$, the Hamiltonian operator H^{PCW}, and the spectral equation are given by

$$\mathfrak{F}^{\mathrm{PCW}} = \{|n\rangle \mid n \in \mathbb{N}\}, \tag{34}$$

$$H^{\mathrm{PCW}} = -\frac{\hbar^2}{2m}\nabla^2 + V^{\mathrm{PCW}},$$
$$H^{\mathrm{PCW}}\Psi^{\mathrm{PCW}} = E^{\mathrm{PCW}}\Psi^{\mathrm{PCW}}, \tag{35}$$

where again the bi-dimensional Laplacian operator is written on cylindrical coordinates. The substitution of (34) with (33) on (35) gives the following partial differential equation whose solution determines the spectrum of H^{PCW}

$$-\frac{\hbar^2}{2m}\nabla^2\Psi^{\mathrm{PCW}} + V^{\mathrm{PCW}}\Psi^{\mathrm{PCW}} = E^{\mathrm{PCW}}\Psi^{\mathrm{PCW}}. \tag{36}$$

Using again cylindrical coordinates and axial symmetry (36) is reduced to

$$\frac{\partial^2}{\partial r}\Psi^{\mathrm{PCW}}(r) + \frac{1}{r}\frac{\partial}{\partial r}\Psi^{\mathrm{PCW}}(r) + \frac{2m}{\hbar^2}\left(E^{\mathrm{PCW}} - W^{\mathrm{PCW}} - \frac{U^{\mathrm{PCW}}}{r^2}\right)\Psi^{\mathrm{PCW}}(r) = 0. \tag{37}$$

The condition of trapping for the particle to the interior of the PCW but with a infinite potential at $r = 0$, is introduced using the boundary conditions

$$\Psi^{\mathrm{PCW}}(R) = 0, \tag{38}$$

$$\Psi^{\mathrm{PCW}}(0) = 0, \tag{39}$$

then the solution of (37) with the condition of wavefunction finite at $r = 0$ is given by

$$\Psi^{\mathrm{PCW}}(r) = C J_{\sqrt{\frac{2mU^{\mathrm{PCW}}}{\hbar^2}}}\left(\sqrt{\frac{2m(E^{\mathrm{PCW}} - W^{\mathrm{PCW}})}{\hbar^2}} r\right), \tag{40}$$

where again C is a constant. Now to obtain in (40) the condition (38), it is necessary that

$$\sqrt{\frac{2mU^{\mathrm{PCW}}}{\hbar^2}} \geqslant 1, \text{ where we have chosen that } \sqrt{\frac{2mU^{\mathrm{PCW}}}{\hbar^2}} = 1. \tag{41}$$

With (41), then (40) changes to

$$\Psi^{\mathrm{PCW}}(r) = CJ_1\left(\sqrt{\frac{2m(E^{\mathrm{PCW}} - W^{\mathrm{PCW}})}{\hbar^2}}r\right). \tag{42}$$

Using (42) and the boundary condition (39), the energy spectrum is obtained as

$$E_n^{\mathrm{PCW}} = W^{\mathrm{PCW}} + \frac{\hbar^2}{2mR^2}\alpha_n^2, \tag{43}$$

where $n \in \mathbb{N}$ and the α_n's are the zeros of the Bessel function $J_1(x)$. Using the empirical formula of interpolation for α_n

$$\alpha_n = 3.14n + 3.83, \tag{44}$$

the substitution of (44) on (43) gives

$$\begin{aligned} E_n^{\mathrm{PCW}} &= W^{\mathrm{PCW}} + \frac{7.34\hbar^2}{mR^2} + \frac{4.93\hbar^2}{mR^2}n(n+2.43) \\ &= \hbar\omega^{\mathrm{PCW}}e_n^{\mathrm{PCW}}, \end{aligned} \tag{45}$$

where $W^{\mathrm{PCW}} = -7.34\hbar^2/mR^2$, $\omega^{\mathrm{PCW}} = 4.93\hbar/mR^2$ and $e_n^{\mathrm{PCW}} = n(n+2.43)$, $n \in \mathbb{N}$.

Finally the normalized wavefunction is

$$\Psi_n^{\mathrm{PCW}}(r) = \frac{1}{R\sqrt{\pi}}\frac{J_1\left(\frac{\alpha_n}{R}r\right)}{J_0(\alpha_n)} \equiv \langle r \mid n\rangle,$$

and the action of H^{PCW} over the space $\mathfrak{F}^{\mathrm{PCW}}$ is given by

$$H^{\mathrm{PCW}}|n\rangle = \hbar\omega^{\mathrm{PCW}}e_n^{\mathrm{ICW}}|n\rangle.$$

To avoid a very heavy notation, we define by the rest of this subsection a new variable denoted by i that can take the values ICW and PCW, that is to say

$$i \in \{\mathrm{ICW}, \mathrm{PCW}\}.$$

With the aim to establish that the $\mathfrak{su}(1,1)$ algebra is the dynamical algebra associated both with the ICW as to the PCW, we follow the procedure that was presented in [17] for the case of ISW, and we introduce both a creation operator denoted by K_+^i as a destruction operator denoted by K_-^i, in this way that we can to rewrite the Hamiltonian H^i as

$$H^i = \hbar\omega^i\left(K_+^iK_-^i\right).$$

Besides of this, we introduce the operator

$$K_3^i = \left[K_-^i, K_+^i\right],$$

in such form that the operators K_+^i, K_-^i, and K_3^i satisfy the commutation relations (14*b*).

With the aim to satisfy the requirements, based on (32) and (45) we establish a representation of the $\mathfrak{su}(1,1)$ algebra, which is given by

$$\begin{aligned} K_-^i|0\rangle &= 0, & K_-^i|n\rangle &= \sqrt{e_n^i}|n-1\rangle, \\ K_+^i|n\rangle &= \sqrt{e_{n+1}^i}|n+1\rangle, & K_3^i|n\rangle &= \left(e_{n+1}^i - e_n^i\right)|n\rangle. \end{aligned}$$

From (32) and (45) we define

$$b^{\mathrm{ICW}} = 1.54 \qquad \text{and} \qquad b^{\mathrm{PCW}} = 2.43,$$

and based on the representation that was introduced, we get a number operator of the form

$$N^i = (1/2)\left(K_3^i - (b^i + 1)\right), \qquad N^i|n\rangle = n|n\rangle.$$

The BGCS for the ICW and for the PCW are the states denoted by $|z\rangle^i$ that satisfy the equation $K_-^i|z\rangle^i = z|z\rangle^i$ and that have the form

$$|z\rangle^i = \frac{|z|^{(b^i)/2}}{\sqrt{I_{b^i}(2|z|)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!(b^i+n)!}}|n\rangle,$$

being the associated probability density denoted by $P_n^i(z)$

$$P_n^i(z) = \frac{|z|^{b^i}}{I_{b^i}(2|z|)} \frac{|z|^{2n}}{n!(b^i+n)!}.$$

Then, for the systems ISW/PCW, we have established that they satisfy the algebraic structure of (14) with

$$f^i(n) = e_n^i \qquad h^i(N^i) = \mathbb{1},$$

and by the use of a procedure that is similar to the realized for the ISW, we obtain an adaptation of the KHQA over the systems ICW/PCW where the halting criterion (6) is satisfied for the values of $z \in \mathbb{C}$ such that

$$P_0^i = \frac{|z|^{b^i}}{(b^i)!I_{b^i}(2|z|)} < 1/2.$$

4.3. The Pöschl–Teller potentials

In this subsection it is showed that the Pöschl–Teller potentials (PTP) also satisfy the algebraic structure given in (14) and then it is possible to adapt the KHQA for the case of the PTP. The problem is to find both the energy spectrum as the wavefunctions for a particle of mass m that is confined within an ISW, is generalized to the case when the particle is trapped by a potential of the Pöschl–Teller kind [17]

$$V_{\lambda,\kappa}^{\mathrm{PTP}}(x) = \frac{1}{2}V_0^{\mathrm{PTP}}\left(\frac{\lambda(\lambda-1)}{\cos^2 x/2l} + \frac{\kappa(\kappa-1)}{\sin^2 x/2l}\right),$$

where the parameters $\lambda, \kappa > 1$, the coupling constant is $V_0 > 0$, and the PTP is defined inside the domain $0 \leqslant x \leqslant \pi l$. The corresponding Hamiltonian is given by

$$H^{\mathrm{PTP}} = i^2\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\hbar^2}{8ml^2}\left(\frac{\lambda(\lambda-1)}{\cos^2 x/2} + \frac{\kappa(\kappa-1)}{\sin^2 x/2}\right) - \frac{\hbar^2}{8ml^2}(\lambda+\kappa)^2, \tag{46}$$

where we chose by convenience $V_0 = \hbar^2/4ml^2$ [17]. The energy spectrum and the corresponding eigenstates come from the solutions of the spectral equation

$$H^{\mathrm{PTP}}\psi^{\mathrm{PTP}}(x) = E^{\mathrm{PTP}}\psi^{\mathrm{PTP}}(x), \tag{47}$$

with the boundary conditions

$$\psi^{\mathrm{PTP}}(0) = \psi^{\mathrm{PTP}}(\pi l) = 0. \tag{48}$$

Replacing (46) on (47) together with the boundary conditions (48), we obtain the normalized wavefunctions and their corresponding eigenvalues [17]

$$\Psi_n^{\mathrm{PTP}}(x) = [c_n(\lambda,\kappa)]^{-1/2}\left(\cos\frac{x}{2l}\right)^{\lambda}\left(\sin\frac{x}{2l}\right)^{\kappa}{}_2F_1\left(-n, n+\lambda+\kappa; \kappa+\frac{1}{2}; \sin^2\frac{x}{2a}\right)$$
$$\equiv \langle x \mid \eta, n\rangle, \qquad E_n^{\mathrm{PTP}} = \hbar\omega^{\mathrm{PTP}} e_n^{\mathrm{PTP}}(\lambda,\kappa),$$

where $[c_n(\lambda,\kappa)]^{-1/2}$ is the normalization factor that is given analytically when λ and κ are positive integers, the function ${}_2F_1$ is a particular case of the generalized hypergeometric function, and

$$\omega^{\mathrm{PTP}} = \frac{\hbar}{2ml^2}, \qquad e_n^{\mathrm{PTP}}(\lambda,\kappa) = n(n+2\eta-1), \qquad \eta = \frac{\lambda+\kappa+1}{2}. \tag{49}$$

The action of the Hamiltonian H^{PTP} over the Fock space defined as

$$\mathfrak{F}^{\mathrm{PTP}} = \{|\eta, n\rangle \mid n \in \mathbb{N}\},$$

is given by

$$H^{\mathrm{PTP}}|\eta, n\rangle = E_n^{\mathrm{PTP}}|\eta, n\rangle.$$

Due to the spectral structure of the PTP, its dynamical algebra is again $\mathfrak{su}(1,1)$ [17], whose generators, denoted now by K_+^{PTP}, K_-^{PTP} and K_3^{PTP}, satisfy the commutation relations (14*b*). Based on (49), the $\mathfrak{su}(1,1)$ algebra admits an infinite-dimensional UIR over the space $\mathfrak{F}^{\mathrm{PTP}}$, which is given by

$$\begin{aligned}
K_-^{\mathrm{PTP}}|\eta,0\rangle &= 0,\\
K_-^{\mathrm{PTP}}|\eta,n\rangle &= \sqrt{e_n^{\mathrm{PTP}}}|\eta,n-1\rangle = \sqrt{n(n+2\eta-1)}|\eta,n-1\rangle,\\
K_+^{\mathrm{PTP}}|\eta,n\rangle &= \sqrt{e_{n+1}^{\mathrm{PTP}}}|\eta,n+1\rangle = \sqrt{(2\eta+n)(n+1)}|\eta,n+1\rangle,\\
K_3^{\mathrm{PTP}}|\eta,n\rangle &= \left(e_{n+1}^{\mathrm{PTP}} - e_n^{\mathrm{PTP}}\right)|\eta,n\rangle = (\eta+n)|\eta,n\rangle.
\end{aligned} \tag{50}$$

Based on the representation of the $\mathfrak{su}(1,1)$ algebra, the Hamiltonian H^{PTP} is rewritten as

$$H^{\mathrm{PTP}} = \hbar\omega^{\mathrm{PTP}}\left(K_+^{\mathrm{PTP}}K_-^{\mathrm{PTP}}\right),$$

and we can construct a new number operator of the form

$$N^{\mathrm{PTP}} = (1/2)\left(K_3^{\mathrm{PTP}} - \eta\right), \qquad N^{\mathrm{PTP}}|\eta,n\rangle = n|\eta,n\rangle.$$

The existence of the dynamical algebra permits the construction of generalized coherent states to $\mathfrak{su}(1,1)$. The state $|\eta,z\rangle^{\mathrm{PTP}}$, $z \in \mathbb{C}$, is chosen again as of the Barut–Girardello type; and again is defined as $K_-^{\mathrm{PTP}}|\eta,z\rangle^{\mathrm{PTP}} = z|\eta,z\rangle^{\mathrm{PTP}}$. As is well known, this is a natural generalization of the coherent state associated with the harmonic oscillator. The explicit form is

$$|\eta,z\rangle^{\mathrm{PTP}} = \left(\Gamma(\eta)\,|z|^{-(\eta-1)}\,I_{\eta-1}(2\,|z|)\right)^{-1/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!(\eta)_n}}|\eta/2,n\rangle,$$

where $(\eta)_n$ is the Pochammer's symbol, defined as $(\eta)_n = \eta(\eta+1)\cdots(\eta+n-1)$; and $I_{\eta-1}$ is again the modified Bessel function of the first class. Similarly to the previous systems, we have a probability density associated with the coherent state that is immediately extracted from the explicit form of the coherent state.

We have established then that the PTP also satisfy the algebraic structure given by (14) with characteristic functions of the form

$$f^{\mathrm{PTP}}(n) = e_n^{\mathrm{PTP}}(\lambda,\kappa) \qquad h^{\mathrm{PTP}}(N^{\mathrm{PTP}}) = \mathbb{1},$$

and by a procedure that is similar to the realized for the cases of ISW, ICW and PCW, we obtain an adaptation of the KHQA for the case of the PTP where it is possible to choose the values of the parameters in such a way that halting criterion (6) is satisfied.

4.4. The density-dependent Holstein–Primakoff system

In this subsection, it is shown that the density-dependent Holstein–Primakoff (HP) system of quantum optics also satisfies the algebraic structure of (14). The HP realization of the Lie algebra $\mathfrak{su}(1,1)$ is constructed from the generators, denoted now by K_+^{HP}, K_-^{HP} and K_3^{HP}, that satisfy the commutation relations (14*b*) [15]. The HP realization explicitly reads

$$K_+^{\mathrm{HP}} = a^+\sqrt{N^{\mathrm{SHO}} + 2\eta}, \qquad K_-^{\mathrm{HP}} = \sqrt{N^{\mathrm{SHO}} + 2\eta}\,a, \qquad K_3^{\mathrm{HP}} = N^{\mathrm{SHO}} + \eta,$$

where a^+, a and N^{SHO} are respectively the creation, annihilation and number operator of a single-mode electromagnetic field and which are given by (1*b*) and (1*f*), and the parameter denoted by η is the label of this representation.

The action of the generators over the Fock space is of the form

$$K_-^{\mathrm{HP}}|0\rangle = 0, \qquad K_-^{\mathrm{HP}}|n\rangle = \sqrt{n(n+2\eta-1)}|n-1\rangle,$$
$$K_+^{\mathrm{HP}}|n\rangle = \sqrt{(2\eta+n)(n+1)}|n+1\rangle, \qquad K_3^{\mathrm{HP}}|n\rangle = (\eta+n)|n\rangle,$$

which is very similar to the representation (50).

The principal difference between HP and the other systems (ISW, ICW, PCW and PTP) is that to the HP is more natural the Perelomov coherent states that the BGCS. Here we consider the Perelomov coherent state as a case of nonlinear coherent states. Then, the equation that defines the nonlinear coherent state that naturally arises for the HP system is [15]

$$\frac{1}{N+2k}K_-^{\mathrm{HP}}|z\rangle^{\mathrm{HP}} = z|z\rangle^{\mathrm{HP}}, \tag{51}$$

where the explicit solution of (51) is

$$|z\rangle^{\mathrm{HP}} = (1-|z|^2)^{M/2}\sum_{n=0}^{\infty}\binom{M+n-1}{n}^{1/2} z^n|n\rangle.$$

Then, we have established that the HP also satisfies the algebraic structure (14) with characteristic functions of the form $f^{\mathrm{HP}}(n) = n(2k+n-1)$ and $h^{\mathrm{HP}}(N^{\mathrm{HP}}) = 1/(N+2k)$ where $N^{\mathrm{HP}} = N^{\mathrm{SHO}}$. It is possible then to adapt the KHQA for the case of the HP with a clearly established halting criterion.

4.5. Laguerre oscillator

Finally, in this subsection, we show that the named Laguerre oscillator (LG) also satisfies the algebraic structure that is given by (14) and then it is possible with such a system to adapt the KHQA. The relevant formalism is the following.

We consider a Hilbert space whose elements are generalized Laguerre functions. By constructing, raising and lowering operators acting on these states, one can obtain an explicit realization of the Hamiltonian, which is defined to be diagonal in this Hilbert space. The obtained system, as defined by the Hamiltonian, is called an Laguerre oscillator.

Now, as is well known, the Laguerre polynomials are defined as [20]

$$L_n^{\alpha}(x) = \frac{1}{n!}\,\mathrm{e}^x x^{-\alpha}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-x}x^{\alpha+n}),$$

and the generalized Laguerre functions are of the form

$$\psi_n^{\alpha}(x) = \sqrt{\frac{n!x^{\alpha+1}\,\mathrm{e}^{-x}}{(n+\alpha)!}}L_n^{\alpha}(x).$$

Now, we can define the raising operator K_+^{LO} and the lowering operator K_-^{LO} from the generalized Laguerre functions

$$K_+^{\mathrm{LO}}\psi_n^\alpha(x) = \left[-x\frac{\mathrm{d}}{\mathrm{d}x} - \frac{2n+\alpha+1-x}{2}\right]\psi_n^\alpha(x)$$
$$= -\sqrt{(n+1)(n+\alpha+1)}\psi_{n+1}^\alpha(x),$$
$$K_-^{\mathrm{LO}}\psi_n^\alpha(x) = \left[x\frac{\mathrm{d}}{\mathrm{d}x} - \frac{2n+\alpha+1-x}{2}\right]\psi_n^\alpha(x)$$
$$= -\sqrt{n(n+\alpha)}\psi_{n-1}^\alpha(x).$$

The generalized Laguerre functions are the base of a Hilbert space that has the structure of Fock space and as a consequence

$$\psi_n^\alpha(x) = \frac{1}{\sqrt{n!(\alpha+1)_n}}\left(K_+^{\mathrm{LO}}\right)^n\psi_0^\alpha(x).$$

The commutator between the ladder operators of the Laguerre oscillator is given by

$$\left[K_-^{\mathrm{LO}}, K_+^{\mathrm{LO}}\right]\psi_n^\alpha(x) = (2n+\alpha+1)\psi_n^\alpha(x),$$

and then, we can define the operator K_3^{LO}, as

$$K_3^{\mathrm{LO}}\psi_n^\alpha(x) = \tfrac{1}{2}(2n+\alpha+1)\psi_n^\alpha(x).$$

The commutation relations for the three operators of the Laguerre oscillator are

$$\left[K_-^{\mathrm{LO}}, K_+^{\mathrm{LO}}\right] = 2K_3^{\mathrm{LO}}, \left[K_3^{\mathrm{LO}}, K_+^{\mathrm{LO}}\right] = K_+^{\mathrm{LO}}, \left[K_3^{\mathrm{LO}}, K_-^{\mathrm{LO}}\right] = -K_-^{\mathrm{LO}},$$

and we conclude that the Laguerre oscillator realizes a infinite-dimensional UIR of $\mathfrak{su}(1,1)$.

The Hamiltonian for the Laguerre oscillator is

$$H^{\mathrm{LO}}\psi_n^\alpha(x) = K_+^{\mathrm{LO}}K_-^{\mathrm{LO}}\psi_n^\alpha(x) = e_n\psi_n^\alpha(x) = n(n+\alpha)\psi_n^\alpha(x),$$

and the BGCS are defined as is usual, it is to say

$$K_-^{\mathrm{LO}}|z\rangle^{\mathrm{LO}} = z|z\rangle^{\mathrm{LO}},$$

where the solution of (4.5) is again the well-known form

$$|z\rangle^{\mathrm{LO}} = \frac{|z|^{\alpha/2}}{\sqrt{I_\alpha(2|z|)}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!(n+\alpha)!}}|n\rangle,$$

where

$$|n\rangle = \psi_n^\alpha(x).$$

Then, we have proved that the Laguerre oscillator also satisfies the algebraic structure of (14) with characteristic functions of the form $f(n) = n(n+\alpha)$ and $h(n) = 1$. All this indicates that it is possible to adapt the KHQA to the case of the Laguerre oscillator.

We can observe that the Laguerre oscillator contains as particular cases the systems ISW, ICW, PCW and PTP for different values of the parameter α. From the other side, it may be possible to have a realization of the Laguerre oscillator and its generalizations within the field of quantum optics.

5. Conclusions

- We have identified from an algebraic point of view the conditions to make adaptations of KHQA: a non-compact Lie algebra of low dimension that admits infinite-dimensional irreducible representations with naturally defined ladder operators and generalized coherent states. Based on this result, we made an adaptation of KHQA over the algebra $\mathfrak{su}(1,1)$ because this algebra satisfies these conditions and because this algebra is the dynamical algebra associated with many quantum systems.

- Hilbert's tenth problem is a semi-computable problem by a TM in the sense that if the Diophantine equation (2) has a solution, an exhaustive search on k-tuplas of non-negative integers would find it, but if (2) does not have a solution this search would not finish. In this sense, it is possible to be interpreted ingenuously that KHQA and our adaptation over the algebra $\mathfrak{su}(1,1)$ carry out an infinite search (in a finite time) on every k-tuplas of non-negative integers. However, KHQA and our adaptation do not make an infinite search, because although Hilbert's tenth problem is TM incomputable, this is a finitely refutable problem [35]. That is to say, it is only necessary to make the search on a finite set of non-negative integers, to determine if (2) has a solution or not, although of course, this finite set is TM incomputable.
- A very common misunderstanding in the technical literature is not to make distinctions between the terms 'quantum computation' and 'standard quantum computation' (e.g. [36, 37]). Due to this misunderstanding and due to equivalence in computability terms, between the standard quantum computation and TM computability established by David Deutsch [38][4], the hypercomputation possibility based on quantum computation is rejected. Nevertheless, this situation is erroneous as is demonstrated by the theoretical existence of KHQA and our adaptation over the algebra $\mathfrak{su}(1,1)$.
- Another common misunderstanding is not to make a distinction between quantum adiabatic computation on finite- and infinite-dimensional Hilbert spaces. For example, there is a recent proof that quantum adiabatic computation is equivalent to standard quantum computation [40]; however, this proof generates no contradiction with KHQA or with our adaptation over the algebra $\mathfrak{su}(1,1)$, because such a proof of equivalence is only valid for quantum adiabatic computation on finite-dimensional Hilbert spaces.
- Based on our adaptation of KHQA over the algebra $\mathfrak{su}(1,1)$, we had presented a plausible realization within the field of condensed matter physics and quantum optics. Although Kieu has refuted successfully some critics of his algorithm (see section *Notes addes* of [12]), there is an important observation with respect to its possible implementation that has not been solved yet, in Kieu's words [13, p 180]:

  > ... there have been some concerns (this pointed has been raised on separate occasions by Martin Davis (2003), Stephen van Enk (2004) and Andrew Hodges (2004)) that infinite precision is still required in physically setting up the various integers parameters in the time-dependent quantum Hamiltonians. While the issue deserves further investigations as surely any systematic errors in the Hamiltonians would be fatal, we still are not convinced that such integer parameters cannot be satisfactorily set up. In particular, we would like to understand the effects of statistical (as opposed to systematic) errors on the statistical behaviour of the spectrum of our adiabatic Hamiltonians.

  This observation is valid for our plausible realization too; however, we agree it is necessary to research further to establish if it is possible or not to implement KHQA or our adaptation over the algebra $\mathfrak{su}(1,1)$.

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[4] In a strict sense there is a type of *weak hypercomputation* in standard quantum computation: the generation of truly random numbers [38]. Nevertheless, it is not clear how to use this property to solve to a TM incomputable problem [39].

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Appendix

In this appendix, we present the procedure to obtain the explicit form of the coherent states denoted by $|z\rangle^{\mathrm{S}}$, which is given by (14*h*) and which was used to obtain the explicit forms for all particular coherent states, both of the Barut–Girardello as well as the Perelomov type, used in this paper. Since the coherent state $|z\rangle^{\mathrm{S}}$ belongs to the Fock space $\mathfrak{F}^{\mathrm{S}}$ (14*a*), we can write the coherent state as an linear combination

$$|z\rangle^{\mathrm{S}} = \sum_{n=0}^{\infty} C_n(z)|n\rangle. \tag{A.1}$$

The substitution of (A.1) on (14*g*) and using (14*c*) and (14*f*) generates the following recurrence equation for the coefficients $C_n(z)$

$$C_{n+1}(z)h^{\mathrm{S}}(n)\sqrt{f^{\mathrm{S}}(n+1)} = zC_n(z). \tag{A.2}$$

The solution of (A.2) is

$$C_n(z) = C_0(z)\frac{z^n}{\left(\prod_{j=0}^{n-1} h^{\mathrm{S}}(j)\right)\left(\sqrt{f^{\mathrm{S}}(n)!}\right)}. \tag{A.3}$$

To obtain the coefficient $C_0(z)$ we apply the condition of normalization of the coherent state

$${}^{\mathrm{S}}\langle z \mid z\rangle^{\mathrm{S}} = 1 = \sum_{n=0}^{\infty} C_0(z)^2 \frac{|z|^{2n}}{\left(\prod_{j=0}^{n-1} h^{\mathrm{S}}(j)\right)^2 (f^{\mathrm{S}}(n)!)}. \tag{A.4}$$

From (A.4) we obtain that

$$C_0(z) = \left(\sum_{m=0}^{\infty} \frac{|z|^{2m}}{\left(\prod_{j=0}^{m-1} h^{\mathrm{S}}(j)\right)^2 (f^{\mathrm{S}}(m)!)}\right)^{-1/2}. \tag{A.5}$$

Finally, the substitution of (A.5) on (A.3) and then on (A.1) gives the following explicit form for the $\mathfrak{su}(1,1)$ nonlinear coherent states, which is given by (14*h*).

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